Multilateral bargaining in networks

Multilateral Bargaining in Networks: On the Prevalence of Inefficiencies

Joosung Lee*

October 2015

Abstract

We introduce a noncooperative multilateral bargaining model for network-restricted environments. In each period, a randomly selected proposer makes an offer by choosing 1) a coalition, or bargaining partners, among the neighbors in a given network and 2) monetary transfers to each member in the coalition. If all the members in the coalition accept the offer, then the proposer buys out their network connections and controls the coalition thereafter. Otherwise, the offer dissolves. The game repeats until the grand-coalition forms, after which the player who controls the grand-coalition wins the unit surplus. All the players have a common discount factor.

The main theorem characterizes a condition on network structures for efficient equilibria. If the underlying network is either *complete* or *circular*, an efficient stationary subgame perfect equilibrium exists for *all* discount factors: all the players always try to reach an agreement as soon as practicable and hence no strategic delay occurs. In *any other* network, however, an efficient equilibrium is impossible if a discount factor is greater than a certain threshold, as some players strategically delay an agreement. We also provide an example of a *Braess-like paradox*, in which the more links are available, the less links are actually used. Thus, network improvements may decrease social welfare.

keywords: noncooperative bargaining, coalition formation, network restriction, buy-out, Braess's Paradox

JEL Classification: C72, C78; D72, D74, D85

1 Introduction

Network restrictions are imposed to multilateral bargaining problems, where an agreement among three or more players is required to generate a unit surplus.[1] To analyze strategic

*Business School, University of Edinburgh, 29 Buccluech Place, Edinburgh EH8 9JS, United Kingdom. E-mail: joosung.lee@ed.ac.uk. This paper constitutes the second chapter of my Ph.D. dissertation submitted to the Pennsylvania State University. This paper has been accepted as a regular paper at the 11th Conference on Web and Internet Economics (WINE 2015) and a one-page abstract will appear in the proceedings of the conference published by Springer-Verlag in the ARCoSS/LNCS series. I am grateful to Kalyan Chatterjee for his guidance, encouragement, and support. I also thank Ed Green, Jim Jordan, Vijay Krishna, Shih En Lu, Neil Wallace, and the participants of 2013(Fall) Midwest Economic Theory Meeting and 20th Coalition Theory Network Workshop for helpful discussions and suggestions. The comments from the three anonymous referees in WINE 2015 were particularly helpful. All remaining errors are mine.

[1] Since Aumann and Dreze (1974), cooperation restrictions have been studied mainly in cooperative games. Myerson (1977) uses a network to described the structure of cooperation restrictions.

behaviors in such environments, we introduce a new noncooperative bargaining model in which each player can bargain only with the directly connected players in a given network. In each period, a proposer is randomly selected and the proposer makes an offer specifying a coalition among the neighbors and monetary transfers to each member in the proposed coalition. If all the members in the coalition accept the offer, then the coalition forms and the proposer buys out the other members' network connections and controls the coalition thereafter. Otherwise, the offer dissolves. The game repeats until the grand-coalition forms, after which the player who controls the grand-coalition wins the unit surplus. All the players have a common discount factor.

The main result characterizes a condition on network structures for efficient equilibria. If the underlying network is either *complete* or *circular*, then for *any* discount factor there exists an efficient stationary subgame perfect equilibrium. In such an efficient equilibrium, all the players always try to reach an agreement as soon as practicable and hence no strategic delay occurs. In any other network, however, an efficient stationary subgame perfect equilibrium is impossible if a discount factor is greater than a certain threshold level, that is, strategic delay must occur at least some positive probability.

We also provide an interesting example in which adding a new communication link decreases social welfare. This observation is reminiscent of the *Braess's paradox* (Braess, 1968). The Braess's paradox refers a situation that constructing a new route reduces overall performance when players choose their route selfishly. Analogously in our model, the more links are available, the less links are actually used, as each player strategically chooses communication links to use for bargaining. As the result, network improvements decrease social welfare.

The model has two important features which distinguish it from the existing noncooperative bargaining models in networks. First, we allow *strategic coalition formation* so that each player can choose the partners to bargain with. In the literature, however, players' strategic interaction is limited in a randomly selected meeting. A bilateral meeting (Manea, 2011a,b; Abreu and Manea, 2012a,b) or a multilateral meeting (Nguyen, 2015) randomly occurs, then the players in the random meeting bargain over their joint surplus.[2] As Hart and Mas-Colell (1996) pointed out, however, a random-meeting model does not entirely capture players' strategic behaviors and strategic decision on coalition formation should also be considered.

Next, we allow players to *buy out* other players and it enables them to gradually form a coalition.In the Manea/Abreu-Manea/Nguyen model, all the players in a randomly selected coalition, once they reach an agreement, must exit the game and they are excluded in further bargaining. Thus players' strategic decision is limited on how to split the coalitional surplus, and hence those models are not applicable to an environment in which gradual coalition formation is inevitable to generate a surplus. On the other hand, when players can buy out other players as an intermediate bargaining step, they not only consider the surplus of the current coalition itself, but also take into account the subsequent bargaining games. Thus players may even form a zero-surplus coalition strategically.

The notion of buyout in bargaining was initially introduced by Gul (1989). In his model multilateral bargaining can be done only through a sequence of random bilateral meetings, and hence players can bargain with only one partner at a time and they cannot choose their bargaining partner. To complement random-meeting models, Lee (2015) allows players to strategically choose their bargaining partners and analyzes players' strategic alliance behaviors in bargaining for general transferable utility environments.

[2] Abreu and Manea (2012a) also consider an alternative model in which a proposer chooses a bargaining partner. However, their model is still limited to a bilateral bargaining.

Unlike in Lee (2015), this paper considers network restrictions in bargaining but concentrates on unanimity-game situations in which only a grand-coalition generates a surplus. There are at least two reasons why unanimity games should be particularly considered for bargaining in networks. First, analyzing unanimity games is enough to show the prevalence of inefficiencies. If any of proper subcoalitions generates a partial surplus, an efficient equilibrium is impossible even in complete networks for high discount factors, as Lee (2015) shows. Second, in unanimity games we can investigate the role of network structure on strategic delay controlling network-irrelevant factors.

The paper is organized as follows. In Section 2, we introduce a noncooperative multilateral bargaining model for a network-restricted environment. Section 3 provides the main characterization result with leading examples. Based on the examples, we discuss a *Braess-Like paradox* in Section 4. Concluding remarks follows in Section 5. The proofs are presented in Appendices.

2 A Model

2.1 Networks

A *network* (or a graph) $g = (N, E)$ consists of a finite set $N = \{1, 2, \cdots, n\}$ of *player*s (or nodes) and a set E of *link*s (or edges) of N. When $g = (N, E)$ is not the only network under consideration, the notations $N(g)$ and $E(g)$ are occasionally used for the player set and the link set rather than N and E to emphasize the underlying network g. Through this paper, we assume that g is simple[3] and connected. Given $g = (N, E)$ and $S \subseteq N$, a *subgraph* restricted on S is $g_{|S} = (S, \{ij \in E \mid \{i, j\} \subseteq S\})$. The (closed) *neighborhood* of $i \in N$ is given by $N_i(g) \equiv \{j \in N \mid \exists ij \in E\} \cup \{i\}$. Let $\deg_i(g) \equiv |N_i(g)| - 1$ be a *degree* of i and $d(i, j; g)$ be a (geodesic) *distance* between i and j in g.

A set $S \subseteq N$ is *dominating* in g if, for all $i \in N$, either $i \in S$ or there exists $j \in S$ such that $ij \in E$. A player $i \in N$ is *dominating* in g if $\{i\}$ is a dominating set. Let $D(g)$ be a set of dominating players in g. A dominating set S is *minimal* if no proper subset is a dominating set. A network is *trivial* if $|N(g)| = 1$. For any integer $k = 2, \cdots, n-1$, a network is *k-regular* if $\deg_i(g) = k$ for all $i \in N(g)$. A network g is *complete* if it is $(n-1)$-regular, or equivalently if $D(g) = N(g)$. A connected network g is *circular* if it is 2-regular.[4]

2.2 A Noncooperative Bargaining Game

A *noncooperative bargaining game*, or shortly a *game*, is a triple $\Gamma = (g, p, \delta)$, where g is a underlying network, $p \in \mathbb{R}^{|N|}_{++}$ is an initial *recognition probability* with $\sum_{i \in N} p_i = 1$, and $0 < \delta < 1$ is a common discount factor.

A game $\Gamma = (g, p, \delta)$ proceeds as follows. In each period, one of the players is randomly selected as a proposer according to p. Then, the proposer i makes an offer, that is, i strategically chooses a pair (S, y) of a coalition $S \subseteq N_i(g)$ and monetary transfers $y \in \mathbb{R}^S_+$ with $\sum_{j \in S} y_j = 0$. Each respondent $j \in S \setminus \{i\}$ sequentially either accepts the offer or rejects it.[5] If any $j \in S \setminus \{i\}$ rejects the offer, then the offer dissolves and all the players

[3] A simple network is an unweighted and undirected network without loops or multiple edges.

[4] A *circular network* (or a *circle*) should not be confused with a *cycle in a network*. A circular network is a network that consists of a single cycle.

[5] The result does not depend on the order of responses.

repeat the same game in the next period. If each $j \in S \setminus \{i\}$ accepts the offer, then i *buys out* $S \setminus \{i\}$, that is, each respondent $j \in S \setminus \{i\}$ leaves the game with receiving y_j from the proposer i and the remaining players $(N \setminus S) \cup \{i\}$ play the subsequent game $\Gamma^{(i,S)}$ in the next period. All the players have a common discount factor δ.

After i buys out $S \setminus \{i\}$, or i forms S, the subsequent game $\Gamma^{(i,S)} = \big(g^{(i,S)}, p^{(i,S)}, \delta\big)$ is defined in the following way:

i) The induced network $g^{(i,S)} = \big(N^{(i,S)}, E^{(i,S)}\big)$, where $N^{(i,S)} = (N \setminus S) \cup \{i\}$ and

$$E^{(i,S)} = \{ij \mid (\exists i'j \in E)\ i' \in S \text{ and } j \in N \setminus S\} \bigcup \{jk \mid (\exists jk \in E)\ j,k \in N \setminus S\}.$$

That is, after i's S-formation, $S \setminus \{i\}$ leaves the network, but i inherits all the network connections from S.

ii) The induced recognition probability $p^{(i,S)}$:

$$p_j^{(i,S)} = \begin{cases} p_S & \text{if } j = i \\ p_j & \text{if } j \in N \setminus S \\ 0 & \text{if } j \in S \setminus \{i\}. \end{cases}$$

That is, the proposer i takes the respondents' chances of being a proposer as well.

The game continues until only one player remains, after which the last player acquires one unit of surplus. When the game ends in finite period T, the history h specifies a finite sequence $\tilde{y}(h) = \{y^t(h)\}_{t=0}^{T}$ of monetary transfers and the last player $i^*(h) \in N$. Given $\Gamma = (g, p, \delta)$ and a history h, player i's discounted sum of expected payoffs is

$$U_i(h) = \sum_{t=0}^{T} \delta^t y_i^t(h) + \delta^T \mathbb{1}(i = i^*(h)).$$

If the game does not end within finite periods, then the history h induces a sequence $\tilde{y}(h)$ of monetary transfers without determining the last player, and hence player i's discounted sum of expected payoffs is

$$U_i(h) = \sum_{t=0}^{\infty} \delta^t y_i^t(h).$$

2.3 Coalitional States

A (coalitional) *state* π is a partition of N, specifying a set of *active player*s $N^\pi \subseteq N$. For each active player $i \in N^\pi$, i's partition block $[i]_\pi$ represents the players i together with players whom he has previously bought out. Denote π° by the initial state, that is, $N^{\pi^\circ} = N$ and $[i]_{\pi^\circ} = \{i\}$ for all $i \in N$. A state π is *terminal* if $|N^\pi| = 1$.

A state π is *feasible* in g, if there exists a sequence of coalition formations $\{(i_\ell, S_\ell)\}_{\ell=1}^{L}$ such that $i_1 \in N$ and $S_{i_1} \subseteq N_{i_1}$; and $i_\ell \in N^{(i_1,S_1)\cdots(i_{\ell-1},S_{\ell-1})}$ and $S_\ell \subseteq N_{i_\ell}^{(i_1,S_1)\cdots(i_{\ell-1},S_{\ell-1})}$ for all $\ell = 2, \cdots, L$; and $N^\pi = N^{(i_1,S_1)\cdots(i_L,S_L)}$. Let $\Pi(g)$ be a set of all feasible states in g. For each $\pi \in \Pi(g)$, the induced network $g^\pi = (N^\pi, E^\pi)$ is uniquely determined by

$$E^\pi \equiv \bigcup_{i \in N^\pi} \Big\{ij \mid \exists i'j' \in E\ (i' \in [i]_\pi \text{ and } j' \in [j]_\pi)\Big\},$$

and the induced recognition probability p^π is determined by

$$p_i^\pi = \begin{cases} \sum_{j \in [i]_\pi} p_j & \text{if } i \in N^\pi \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

When there is no danger of confusion, we omit π° in notations, for instance, $g^{\pi^\circ} = g$, $g^{\pi^\circ(i,S)} = g^{(i,S)}$, and so on. The description of the underlying network g may also be omitted, when it is clear. For notational simplicity, for any $v \in \mathbb{R}^{|N|}$ and any $S \subseteq N$, we denote $v_S = \sum_{j \in S} v_j$.

2.4 Stationary Subgame Perfect Equilibria

We focus on stationary subgame perfect equilibria. A stationary strategy depends only on the current coalitional state and within-period histories, but not the histories of past periods. The existence of a stationary subgame perfect equilibrium is known in the literature including Eraslan (2002) and Eraslan and McLennan (2013). See Lee (2015) for the formal description of stationary strategies. In the literature, instead of considering all the possible stationary strategies, a simple stationary strategy, namely a *cutoff strategy*, is usually accepted.

A cutoff strategy profile $(\mathbf{x}, \mathbf{q})$ consists of a *value profile* $\mathbf{x} = \{\{x_i^\pi\}_{i \in N^\pi}\}_{\pi \in \Pi}$ and a *coalition formation strategy profile* $\mathbf{q} = \{\{q_i^\pi\}_{i \in N^\pi}\}_{\pi \in \Pi}$, where $x_i^\pi \in \mathbb{R}$ and $q_i^\pi \in \Delta(2^{N_i^\pi})$ for each $\pi \in \Pi(g)$.[6] A cutoff strategy profile $(\mathbf{x}, \mathbf{q})$ specifies the behaviors of an active player $i \in N^\pi$: in the following way:

- player i proposes (S, y) with probability $q_i^\pi(S)$ such that

$$y_k = \begin{cases} \delta x_k^\pi & \text{if } k \in S \setminus \{i\} \\ -\delta x_{S\setminus\{i\}}^\pi & \text{if } k = i \\ 0 & \text{otherwise;} \end{cases}$$

- player i accepts any offer (S, y) with $i \in S$ if and only if $y_i \geq \delta x_i^\pi$.

Note that player i can decline to make an offer by choosing $S = \{i\}$. A cutoff strategy profile $(\mathbf{x}, \mathbf{q})$ induces a probability measure $\mu_{\mathbf{x},\mathbf{q}}$ on the set of all possible histories. Given history h, let $\tilde{\pi}(h) = \{\pi^t(h)\}_{t=0}^T$ be a sequence of states which is determined by h. Given $(\mathbf{x}, \mathbf{q})$, define the set of *inducible states*:

$$\Pi_{\mathbf{x},\mathbf{q}}(g) = \{\pi \in \Pi(g) \mid (\exists h\ \exists t)\ \mu_{\mathbf{x},\mathbf{q}}(h) > 0 \text{ and } \pi = \pi^t(h)\}.$$

Given $\mathbf{x}$, for each $\pi \in \Pi(g)$, $i \in N^\pi$, and $S \subseteq N_i^\pi$, define a player i's *excess surplus* of S-formation:

$$e_i^\pi(S, \mathbf{x}) = \begin{cases} \delta x_i^{\pi(i,S)} - \delta x_S^\pi & \text{if } S \subsetneq N^\pi \\ 1 - \delta x_{N^\pi}^\pi & \text{if } S = N^\pi. \end{cases}$$

Let $\mathcal{D}_i^\pi(\mathbf{x}) = \text{argmax}_{S \subseteq N_i^\pi} e_i^\pi(S, \mathbf{x})$ be a *demand set* of player i in π and $m_i^\pi(\mathbf{x}) = \max_{S \subseteq N_i^\pi} e_i^\pi(S, \mathbf{x})$ be a (net) *proposal gain* of player i in π. Given a cutoff strategy

[6]Through this paper, for a finite set X, $\Delta(X)$ is the set of all possible probability measures in X.

profile $(\mathbf{x}, \mathbf{q})$, define an active player i's *continuation payoff* in π:

$$
\begin{aligned}
u_i^\pi(\mathbf{x},\mathbf{q}) &= p_i^\pi \sum_{S\subseteq N^\pi} q_i^\pi(S) e_i^\pi(S,\mathbf{x}) + \sum_{j\in N^\pi} p_j^\pi \left(\sum_{S:i\in S\subseteq N^\pi} q_j^\pi(S)\delta x_i^\pi + \delta \left(\sum_{S:i\notin S\subseteq N^\pi} q_j^\pi(S) x_i^{\pi(j,S)} \right)\right) \\
&= p_i^\pi \sum_{S\subseteq N^\pi} q_i^\pi(S) e_i^\pi(S,\mathbf{x}) + \delta \left(\sum_{j\in N^\pi} p_j^\pi \sum_{S\subseteq N^\pi} q_j^\pi(S) \left(\mathbb{1}(i\in S) x_i^\pi + \mathbb{1}(i\notin S) x_i^{\pi(j,S)} \right)\right). \quad (2)
\end{aligned}
$$

We close this section with two important lemmas which provide fundamental tools for our analysis. Lemma 1 shows that any stationary subgame perfect equilibrium can be uniquely represented by a cutoff strategy equilibrium in terms of a equilibrium payoff vector. Thus, when we are interested in players' equilibrium payoffs or efficiency, without loss of generality, we may consider only cutoff strategy equilibria. Through this paper, an equilibrium refers a cutoff strategy equilibrium. Lemma 2 characterizes a cutoff strategy equilibrium with two tractable conditions, *optimality* and *consistency*. More general versions of the proofs can be found in Lee (2015).

Lemma 1. *For any stationary subgame perfect equilibrium, there exists a cutoff strategy equilibrium which yields the same equilibrium payoff vector.*

Lemma 2. *A cutoff strategy profile* $(\mathbf{x}, \mathbf{q})$ *is an stationary subgame perfect equilibrium if and only if, for all* $\pi \in \Pi$ *and* $i \in N^\pi$*, the following two conditions hold,*

i) **Optimality:** $q_i^\pi \in \Delta(\mathcal{D}_i^\pi(\mathbf{x}))$*; and*

ii) **Consistency:** $x_i^\pi = u_i^\pi(\mathbf{x}, \mathbf{q})$.

3 Efficient Equilibria

In this section, we characterize a necessary and sufficient condition on network structures for efficient equilibria. Given g, define a *maximum coalition formation strategy profile* $\bar{\mathbf{q}} = \{\{\bar{q}_i^\pi\}_{i\in N^\pi}\}_{\pi\in\Pi(g)}$ with

$$
\bar{q}_i^\pi(S) = \begin{cases} 1 & \text{if } S = N_i^\pi \\ 0 & \text{otherwise,} \end{cases}
$$

that is, for each state $\pi \in \Pi(g)$, each proposer $i \in N^\pi$ chooses a maximum coalition N_i^π to bargain with. Given $\Gamma = (g, p, \delta)$, let $\bar{u}(\Gamma)$ be a maximum welfare. Note that $\bar{u}(\Gamma)$ is obtained by any cutoff strategy profile involves with a maximum coalition formation strategy profile. A strategy profile $(\mathbf{x}, \mathbf{q})$ is *efficient* if

$$
\sum_{i\in N} u_i(\mathbf{x},\mathbf{q}) = \bar{u}(\Gamma). \quad (3)
$$

Example 1. Let $N = \{1, 2, 3, 4\}$. Consider two game $\Gamma = (g, p, \delta)$ and $\Gamma' = (g', p, \delta)$, where $g = (N, \{12, 23, 34, 41\})$ is a circular network and $g' = (N, \{12, 23, 34, 41, 13\})$ is a chordal network. For any p, it is easy to see $\bar{u}(\Gamma) < \bar{u}(\Gamma')$ as $\bar{u}(\Gamma) = \delta$ and $\bar{u}(\Gamma') = (p_1 + p_3) + \delta(p_2 + p_4)$. □

An efficient strategy profile does not necessarily consist of maximum coalition formation strategies. The following Lemma 3 characterizes the coalition formation strategies which constitute an efficient equilibrium. The proof is presented in Appendix A. For each $\pi \in \Pi(g)$, define a set of i's coalitions which maximizes the sum of players' expected payoffs in the subsequent state:

$$\mathcal{E}_i^\pi \equiv \underset{S \subseteq N_i^\pi}{\operatorname{argmax}} \, \bar{u}(\Gamma^{\pi(i,S)}).$$

Lemma 3. *Given $\Gamma = (g, p, \delta)$, an equilibrium $(\mathbf{x}, \mathbf{q})$ is efficient if and only if,*

$$\forall \pi \in \Pi_{\mathbf{x},\mathbf{q}}(g) \quad \forall i \in N^\pi \quad q_i^\pi \in \Delta(\mathcal{E}_i^\pi).$$

Now we state our main theorem, which characterizes a condition on network structures for efficient equilibria.

Theorem 1. *An efficient stationary subgame perfect equilibrium exists for all discount factors if and only if the underlying network is either* complete *or* circular*.*

The theorem presents that the strategic delay in bargaining is a prevalent phenomena and it causes inefficiency. For any network which is neither complete nor circular, players strategically exclude some of their neighbors from bargaining partners if the discount factor is sufficiently large but strictly less than one. By doing so, they can be in a better position in an induced network and hence increase their future bargaining power, but this inevitably yields social inefficiency. If the underlying network is either complete or circular, on the other hand, coalition formations never change the structure of network – in terms of graph theory, any vertex contraction on adjacent nodes in a complete or a circular network induces a complete or a circular network. Thus, players can not drastically change their position by coalition formation and hence all the players try to reach an agreement as soon as practicable and no strategic delay occurs. Those results do not depend on the players' recognition probability as long as each player has a positive chance of being a proposer.

We prove the theorem through the four propositions. For the sufficient condition, in subsection 3.1, we construct an efficient equilibrium in a complete network (Proposition 1) and in a circular network (Proposition 2). For the necessary condition, Proposition 3 shows the inefficiency result for a specific class of networks, namely *pre-complete networks*, in subsection 3.2. That is, if the underlying network is pre-complete and non-circular, then any stationary subgame perfect equilibrium is inefficient for a sufficiently high discount factor. In subsection 3.3, Proposition 4 completes the necessary condition by showing that, for any game with an incomplete non-circular network, any efficient strategy induces a pre-complete non-circular network with positive probability. In the main body, the propositions are formally stated and leading examples are provided to highlight the underlying insights.

3.1 The Sufficient Condition

First, we consider a complete network. Proposition 1 shows that a unanimous agreement is always immediately reached for any p and δ. Furthermore, the equilibrium payoff vector is unique and hence *any* stationary subgame perfect equilibrium is efficient. Let $\mathbf{p} = \{\{p_i^\pi\}_{i \in N^\pi}\}_{\pi \in \Pi}$.

Proposition 1. *Let g be a complete network. For any $\Gamma = (g, p, \delta)$,*

i) *there exists a cutoff strategy equilibrium* $(\mathbf{p}, \bar{\mathbf{q}})$;

ii) *for any equilibrium, the equilibrium payoff vector equals to* p.

Example 2 (A Three-Player Complete Network)**.** Let g be a complete network with $N(g) = \{i, j, k\}$ and p be an initial recognition probability. In the first period, a proposer i forms a grand-coalition by buying out other two players at the prices of δp_j and δp_k. Thus the unit surplus belongs to i and his payoff is $1 - (p_j + p_k)\delta = 1 - (1 - p_i)\delta$. Thus, player i's expected payoff is $p_i \cdot (1 - (1 - p_i)\delta) + (p_j + p_k) \cdot \delta p_i = p_i$. □

Next, in a circular network, we construct an efficient equilibrium in which each player always forms a maximum coalition and the equilibrium payoff vector is proportional to the initial recognition probability. Recall that $\lfloor x \rfloor$ is the largest integer not greater than x.

Proposition 2. *Let g be a circular network. For any $\Gamma = (g, p, \delta)$, there exists a cutoff strategy equilibrium $(\mathbf{x}, \bar{\mathbf{q}})$, where for all $\pi \in \Pi(g)$ and all $i \in N^\pi$,*

$$x_i^\pi = \delta^{\left\lfloor \frac{|N^\pi|}{2} \right\rfloor - 1} p_i^\pi. \tag{4}$$

Example 3 (A Four-Player Circular Network)**.** Let g be a circular network with $|N(g)| = 4$. For all π with $2 \leq |N^\pi| \leq 3$, since g^π is complete, the equilibrium strategies in a non-initial state π are $x^\pi = p^\pi$ and $q^\pi = \bar{q}^\pi$, which are consistent with (4). For the initial state, take any $i \in N$ and let $N_i = \{i, j, k\}$. For any $\{i\} \subsetneq S \subseteq N_i$, since S-formation induces a complete network, the excess surplus from S-formation is $e_i(S, \mathbf{x}) = p_S\delta - \delta x_S = \delta(1 - \delta)p_S$, which implies $\mathcal{D}_i = \{N_i\}$. For all $\ell \in N$, then $q_\ell(N_\ell) = 1$. Thus, we have $\sum_{\ell \in N} p_\ell \sum_{S \ni i} q_\ell(S) = p_{N_i}$ and $\sum_{\ell \in N} p_\ell \sum_{S \not\ni i} q_\ell(S) = 1 - p_{N_i}$. Therefore, i's expected payoff is:

$$u_i(\mathbf{x}, \bar{\mathbf{q}}) = p_i \cdot \delta(1 - \delta)p_{N_i} + \delta\left[p_{N_i} \cdot \delta p_i + (1 - p_{N_i}) \cdot p_i\right] = \delta p_i,$$

which satisfies consistency condition. □

3.2 The Necessary Condition : Pre-complete Networks

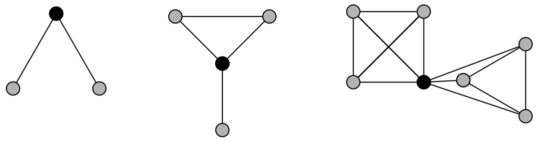
To prove the necessary condition, we will show that any efficient strategy profile cannot constitue an equilibrium in any incomplete non-circular network if the discount factor is sufficiently high. First, we need to define a special class of networks, namely *pre-complete* networks, in which all the players can induce a complete network. Given g, denote a set of i's feasible coalitions which yield a complete network by

$$\mathcal{C}_i(g) = \{S \subseteq N_i(g) \mid g^{(i,S)} \text{ is complete.}\}$$

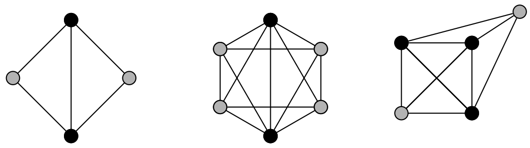
Definition 1. A graph g is *pre-complete* if

$$\forall i \in N(g) \quad \{i\} \notin \mathcal{C}_i(g) \neq \emptyset.$$

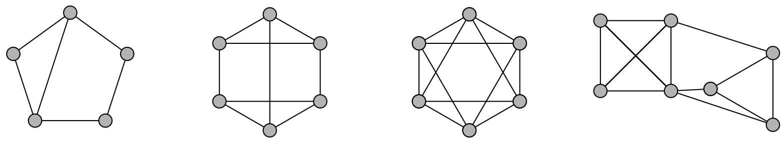
See Figure 1 for examples of pre-complete networks. Pre-complete networks can be distinguished by the four subclasses – (a) networks with a single dominating player, (b) networks with multiple dominating players, (c) networks with no dominating player, and (d) circular networks. In a circular network, as Proposition 2, there exists an efficient equilibrium for any discount factor. Other subclasses of pre-complete networks provide a different insight for strategic delay, we provide three leading examples in the following subsections.



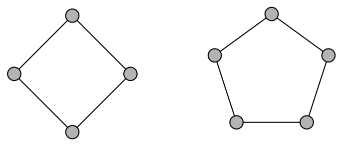
(a) Single Dominating Player: Inefficient



(b) Multiple Dominating Players: Inefficient



(c) No Dominating Player (Non-circular): Inefficient



(d) Circular: Efficient.

Figure 1: Examples of Pre-complete Networks: Dark nodes represent dominating players.

Proposition 3. *Let g be a pre-complete non-circular network. For any p, there exists $\bar{\delta} < 1$ such that for all $\delta > \bar{\delta}$, any efficient strategy profile $(\mathbf{x}, \mathbf{q})$ cannot be an equilibrium in $\Gamma = (g, p, \delta)$.*

3.2.1 Pre-complete Network with a Single Dominating Player

When there is a single dominating player, the other non-dominating players are reluctant to form a coalition with the dominating player. This causes a delay. For instance in a three-player chain, in which there is only one dominating player, the unique dominating player has a stronger bargaining power than the other players so that her value is too high for the other players to buy her out. Thus, the non-dominating players decline to make an offer, even when they are recognized as a proposer.

Example 4 (A Chain)**.** Let $g = (\{1, 2, 3\}, \{12, 13\})$. First, we show an impossibility of an efficient equilibrium. Suppose there exists an efficient equilibrium $(\mathbf{x}, \mathbf{q})$. Then player 1 is always included in a proposed coalition, that is, $q_1(N) = q_2(\{1, 2\}) = q_3(\{1, 3\}) = 1$. Thus player 1's expected payoff is $u_1(\mathbf{x}, \mathbf{q}) = p_1(1 - \delta x_N) + \delta x_1$. Since $x_1 = u_1(\mathbf{x}, \mathbf{q})$ and $x_N = p_1 + (1 - p_1)\delta$, it follows $(1 - \delta)x_1 = p_1(1 - \delta(p_1 + (1 - p_1)\delta))$, or equivalently,

$$x_1 = p_1(1 + (1 - p_1)\delta). \tag{5}$$

On the other hand, player 2's expected payoff is

$$u_2(\mathbf{x}, \mathbf{q}) = p_2 m_2(\mathbf{x}) + \delta((p_1 + p_2)x_2 + p_3 p_2) \geq \delta(1 - p_3)x_2 + p_3 p_2 \delta.$$

By consistency, we have $x_2 \geq \frac{\delta p_2 p_3}{1-\delta(1-p_3)}$ and similarly $x_3 \geq \frac{\delta p_2 p_3}{1-\delta(1-p_2)}$. Together with (5), it requires that

$$x_N \geq p_1(1+(1-p_1)\delta) + \frac{\delta p_2 p_3}{1-\delta(1-p_3)} + \frac{\delta p_2 p_3}{1-\delta(1-p_2)}.$$

To see a contradiction, as δ converges to 1, observe that the right-hand side converges to $1 + p_1(1-p_1)$, which is strictly greater than 1 as long as $p_1 > 0$. However, x_N never exceeds 1. Thus, for a sufficiently high δ, the efficient strategy profile $(\mathbf{x}, \mathbf{q})$ cannot be an equilibrium.

Next, we construct an inefficient equilibrium. Let $\bar{\delta} = \max\left\{\frac{p_2}{(p_1+p_2)(1-p_1)}, \frac{p_3}{(p_1+p_3)(1-p_1)}\right\}$ so that $\bar{\delta} < 1$. Consider a strategy profile $(\mathbf{x}, \mathbf{q})$ such that

- $x_1 = \frac{p_1}{1-(1-p_1)\delta}$; $x_2 = x_3 = 0$; and
- $q_1(N) = q_2(\{2\}) = q_3(\{3\}) = 1$,

and in any two-player subgame the active players follow the strategy according to Proposition 1. Since player 2 and player 3 decline to be a proposer in the initial state, the strategy profile is inefficient. To see that $(\mathbf{x}, \mathbf{q})$ constructs an equilibrium for $\delta > \bar{\delta}$, due to Lemma 2, it suffices to verify the following two conditions.

i) *Optimality:* Calculate each player's excess surpluses. It is easy to see that $e_1(N, \mathbf{x}) > 0$ and $e_i(\{i\}, \mathbf{x}) = 0$ for all $i \in N$. For all $i \in \{1, 2\}$, due to Proposition 1, $x_i^{(i,\{1,2\})} = p_1 + p_2$, and hence

$$\begin{aligned} e_i(\{1,2\}, \mathbf{x}) &= \delta(p_1+p_2) - \delta(x_1+x_2) = \delta(p_1+p_2) - \delta\left(\tfrac{p_1}{1-(1-p_1)\delta} + 0\right) \\ &= \tfrac{\delta}{1-(1-p_1)\delta}\left(p_2 - (p_1+p_2)(1-p_1)\delta\right). \end{aligned}$$

Then, $\delta > \bar{\delta}$ implies $e_i(\{1,2\}, \mathbf{x}) < 0$. Similarly, we have $e_i(\{1,3\}, \mathbf{x}) < 0$ for all $i \in \{1,3\}$. Given $\mathbf{x}$, therefore, $\mathcal{D}_1 = \{N\}$, $\mathcal{D}_2 = \{\{2\}\}$, and $\mathcal{D}_3 = \{\{3\}\}$.

ii) *Consistency:* Compute each player's expected payoff:

- $u_1(\mathbf{x}, \mathbf{q}) = p_1 e(N, \mathbf{x}) + \delta x_1 = p_1(1 - \delta x_1) + \delta x_1 = \frac{p_1}{1-(1-p_1)\delta}$
- $u_2(\mathbf{x}, \mathbf{q}) = p_2 e(\{2\}, \mathbf{x}) + \delta x_2 = p_2 \cdot 0 + \delta \cdot 0 = 0$
- $u_3(\mathbf{x}, \mathbf{q}) = p_3 e(\{3\}, \mathbf{x}) + \delta x_3 = p_3 \cdot 0 + \delta \cdot 0 = 0.$

Therefore, $u_i(\mathbf{x}, \mathbf{q}) = x_i$ for all $i \in N$. □

3.2.2 Pre-complete Network with Multiple Dominating Players

Even if there are multiple dominating players, as see (b) in Figure 1, they can generate an additional advantage by forming a *cut coalition* with other dominating players and splitting non-dominating players into two isolated groups. In the next example, we construct an equilibrium in a chordal network in which there are two dominating players.

Example 5 (A Chordal Network)**.** Let $g = (\{1,2,3,4\}, \{12, 23, 34, 41, 13\})$ and $p = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$. Suppose $\delta > \bar{\delta} \approx 0.91$.[7] We construct an equilibrium $(\mathbf{x}, \mathbf{q})$ such that

[7]Note that $\bar{\delta}$ is a solution to $\delta(8 - 8\delta + \delta^2) = (4-\delta)(1-\delta)(4 + 2\delta - \delta^2)$.

- $x_1 = x_3 = \frac{(6-\delta)\delta}{4(4-\delta)(2-\delta)}$; $x_2 = x_4 = \frac{(6-6\delta+\delta^2)\delta}{4(4-\delta)(2-\delta)}$;
- $q_1(\{1,3\}) = q_3(\{1,3\}) = 1$; $q_2(\{1,2\}) = q_2(\{2,3\}) = q_4(\{1,4\}) = q_4(\{3,4\}) = \frac{1}{2}$.

In any subgame in which the number of active players is less than or equal to three, they follows the equilibrium strategies according to Proposition 1 and Example 4. Note that the equilibrium welfare is $x_N = \frac{\delta(3-\delta)}{2(2-\delta)}$. The equilibrium payoff vector converges to $\left(\frac{5}{12}, \frac{1}{12}, \frac{5}{12}, \frac{1}{12}\right)$ as $\delta \to 1$. Now we verify the equilibrium conditions.

i) *Odd Players' Optimality:* Since $\delta > \frac{3}{4}$, Example 4 implies that $x_1^{(1,\{1,3\})} = \frac{p_1+p_3}{1-(1-p_1-p_3)\delta} = \frac{1}{2-\delta}$ and $x_1^{(1,\{1,3\})} = x_4^{(1,\{1,3\})} = 0$. Given $\mathbf{x}$, calculate player 1's excess surpluses:

- $e_1(\{1,2\},\mathbf{x}) = \delta x_1^{(1,\{1,2\})} - \delta(x_1 + x_2) = \frac{\delta(1-\delta)(4-\delta)}{4(2-\delta)}$
- $e_1(\{1,3\},\mathbf{x}) = \delta x_1^{(1,\{1,3\})} - \delta(x_1 + x_3) = \frac{\delta(8-8\delta+\delta^2)}{2(2-\delta)(4-\delta)}$
- $e_1(\{1,2,4\},\mathbf{x}) = \delta x_1^{(1,\{1,2,4\})} - \delta(x_1 + x_2 + x_4) = \frac{\delta(6-6\delta+\delta^2)}{2(4-\delta)}$
- $e_1(N,\mathbf{x}) = 1 - \delta x_N = \frac{(1-\delta)(4+2\delta-\delta^2)}{2(2-\delta)}$

Given $e_1(S,\mathbf{x})$ for all $S \subseteq N_1$, it is routine to see that $\delta > \bar{\delta}$ implies $\mathcal{D}_1(\mathbf{x}) = \{\{1,3\}\}$. Similarly, we also have $\mathcal{D}_3(\mathbf{x}) = \{\{1,3\}\}$.

ii) *Even Players' Optimality:* For any $\{2\} \subsetneq S \subseteq N_2$, player 2's S-formation induces a complete network. Thus, given $\mathbf{x}$, one can compute player 2's excess surpluses:

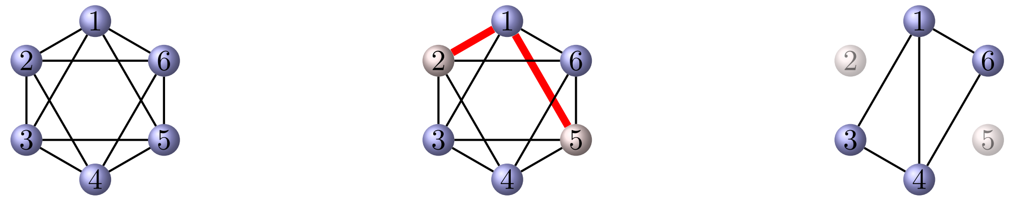
- $e_2(\{1,2\},\mathbf{x}) = e_2(\{2,3\},\mathbf{x}) = \delta x_2^{(2,\{1,2\})} - \delta(x_1 + x_2) = \frac{\delta(1-\delta)(4-\delta)}{4(2-\delta)}$
- $e(\{1,2,3\},\mathbf{x}) = \delta x_2^{(2,\{1,2,3\})} - \delta(x_1 + x_2 + x_3) = \frac{\delta(24-36\delta+11\delta^2-\delta^3)}{4(2-\delta)(4-\delta)}$

Observe that $e_2(\{1,2\},\mathbf{x}) = e_2(\{2,3\},\mathbf{x}) > 0$ for all δ; while $e(\{1,2,3\},\mathbf{x})$ is strictly negative if $\delta > \bar{\delta}$. Thus, for any $\delta > \bar{\delta}$, we have $\mathcal{D}_2(\mathbf{x}) = \{\{1,2\},\{2,3\}\}$ and similarly $\mathcal{D}_4(\mathbf{x}) = \{\{1,4\},\{3,4\}\}$.

iii) *Consistency:* Given $(\mathbf{x},\mathbf{q})$, compute each players' expected payoffs:

- $$\begin{aligned} u_1(\mathbf{x},\mathbf{q}) &= p_1 e(\{1,3\},\mathbf{x}) + \delta \left[\left(p_1 + p_3 + \tfrac{1}{2}(p_2+p_4)\right) x_1 + \tfrac{p_2}{2} x_1^{(2,\{2,3\})} + \tfrac{p_4}{2} x_1^{(4,\{3,4\})} \right], \\ &= \tfrac{1}{4} \cdot \tfrac{\delta(8-8\delta+\delta^2)}{2(2-\delta)(4-\delta)} + \delta \left[\tfrac{3}{4} \cdot \tfrac{(6-\delta)\delta}{4(4-\delta)(2-\delta)} + \tfrac{1}{2} \cdot \tfrac{1}{8} \right] \\ &= \tfrac{(6-\delta)\delta}{4(4-\delta)(2-\delta)} = x_1 \end{aligned}$$
- $$\begin{aligned} u_2(\mathbf{x},\mathbf{q}) &= p_2 e(\{1,2\},\mathbf{x}) + \delta \left[p_2 x_2 + p_4 p_2 + (p_1+p_3) \cdot 0 \right], \\ &= \tfrac{1}{4} \cdot \tfrac{\delta(1-\delta)(4-\delta)}{4(2-\delta)} + \delta \left[\tfrac{1}{4} \cdot \tfrac{(6-6\delta+\delta^2)\delta}{4(4-\delta)(2-\delta)} + \tfrac{1}{4} \cdot \tfrac{1}{4} \right] \\ &= \tfrac{(6-6\delta+\delta^2)\delta}{4(4-\delta)(2-\delta)} = x_2 \end{aligned}$$

and similarly $u_3(\mathbf{x},\mathbf{q}) = x_3$ and $u_4(\mathbf{x},\mathbf{q}) = x_4$, and hence consistency holds. $\square$



(a) 6-player 4-regular network (b) Players form a 3-player coalition (c) A chordal network is induced

Figure 2: Strategic Delay in a Regular Network.

3.2.3 Pre-complete Networks with No Dominating Player

Now we consider a network without a dominating player. See (c) in Figure 1 for instance. Even in a case of that there is no dominating player, we will show that some players can be a dominating player in the induced network by buying out only a part of their neighbors.

Example 6 (A 6-Player 4-Regular Network)**.** Consider a 6-player 4-regular network g as in Figure 2 (a) and let $p_i = \frac{1}{6}$ for all $i = 1, 2, \cdots, 6$. Each proposer can form a 4- or 5-player coalition for an efficient outcome. However, players always form a 3-player coalition in any equilibrium rather than pursuing an efficient outcome. To see this, suppose there exists an efficient equilibrium $(\mathbf{x}, \mathbf{q})$. Since $\bar{u}(g, p, \delta) = \delta$, it is easy to see $x_i = \frac{\delta}{6}$ for all $i = 1, 2, \cdots, 6$. If a player forms a 3-player coalition as in Figure 2 (b), then a chordal network is induced. For the induced game $\Gamma^{(1,\{1,2,5\})}$, for sufficiently high δ, one can construct an equilibrium in which player 1 and player 4 form a cut coalition with each other and player 3 and player 6 form a coalition with one of the connected players as similar in Example 5, and the equilibrium payoffs are:

- $x_1^{(1,\{1,2,5\})} = \frac{-\delta^2+21\delta+18}{6(3-\delta)(6-\delta)}$,
- $x_3^{(1,\{1,2,5\})} = x_6^{(1,\{1,2,5\})} = \frac{\delta(\delta^2-11\delta+12}{6(3-\delta)(6-\delta)}$, and
- $x_4^{(1,\{1,2,5\})} = \frac{-\delta^2+13\delta-6}{2(3-\delta)(6-\delta)}$,

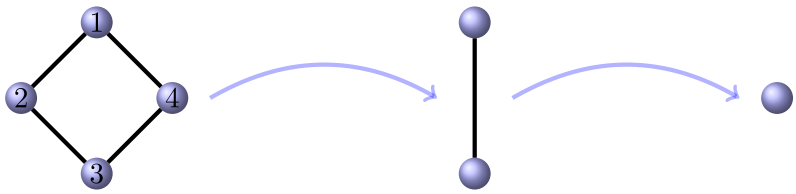
and converge to $\frac{19}{30}$, $\frac{1}{30}$, and $\frac{3}{10}$ as $\delta \to 1$. Going back to the initial game, compare the excess surpluses. For any $S \subset N_1$ with $|S| \geq 4$, player 1's S-formation induces a complete network and hence

$$e_1(S, \mathbf{x}) = \delta x_1^{(1,S)} - \delta x_S = \delta p_S - \delta x_S = \delta(1-\delta)\frac{|S|}{6},$$

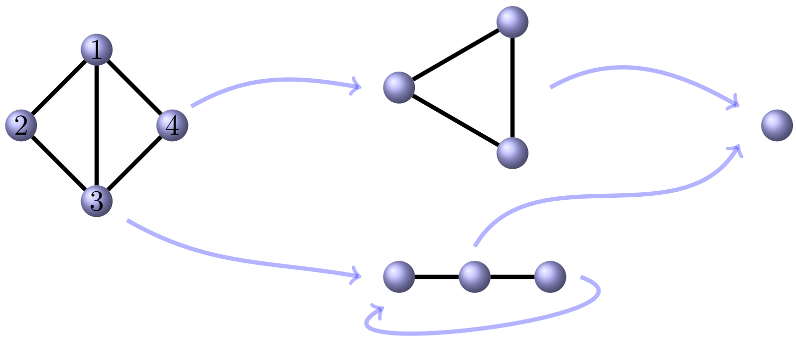
which converges to zero as $\delta \to 1$. On the other hand,

$$e_1(\{1,2,5\}, \mathbf{x}) = \delta x_1^{(1,\{1,2,5\})} - \delta\left(x_1 + x_2 + x_5\right) = \delta\left(\frac{-\delta^2+21\delta+18}{6(3-\delta)(6-\delta)} - \frac{\delta}{3}\right),$$

which converges to $\frac{3}{10}$ as $\delta \to 1$. Thus the optimality condition is violated for sufficiently high δ, and hence an efficient equilibrium is impossible. □



(a) Bargaining in a Circular Network (See Example 3): It takes exactly 2 periods for a grand-coalition in any equilibrium. In the first period, any proposer forms a three-player coalition by buying out two neighbors. Then the induced game is of two players.



(b) Bargaining in a Chordal Network (See Example 5): The expected periods for a grand-coalition is strictly greater than 2. In the first period, if the even players are selected as a proposer, then they choose one of the odd players as a bargaining partner to induce a three-player circle. In the circle, grand-coalition immediately forms. However, if the odd players are initially selected as a proposer, then they induce a three-player chain. In the chain, the leaf players decline to make an offer and hence an additional delay occurs with positive probability.

Figure 3: A Braess-Like Paradox

3.3 The Necessary Condition : Incomplete Networks

We have considered pre-complete non-circular networks. To complete the necessary condition, now we consider a general incomplete non-circular network, beyond the class of pre-complete networks. Proposition 4 implies that for any game with an incomplete non-circular network, if the players play efficient strategies, then a pre-complete non-circular network must be induced with positive probability. Once a pre-complete non-circular network is induced, delay occurs with positive probability by Proposition 3.

Proposition 4. *Let g be an incomplete network. For any efficient strategy profile $(\mathbf{x}, \mathbf{q})$, there exists $\pi \in \Pi_{\mathbf{x},\mathbf{q}}(g)$ such that g^π is a pre-complete network. In addition, if g is a non-circular network, then there exists $\pi \in \Pi_{\mathbf{x},\mathbf{q}}(g)$ such that g^π is a pre-complete non-circular network.*

4 Braess's Paradox

Comparing Example 5 with Example 3, we observe a negative welfare effect of adding a new communication link. In the four-player circle with $p_i = \frac{1}{4}$ for all $i \in N$, the maximum welfare level δ is achieved in an equilibrium. If we add a link between player 1 and player 3 in the circular network, then it becomes a chordal network as in Example 5. Since odd players can form a grand-coalition immediately, the maximum welfare level is $\frac{1}{2}(1 + \delta)$, which is strictly greater than that of the circular network. However, the equilibrium welfare in Example 5 is $\frac{\delta(3-\delta)}{2(2-\delta)}$ which is strictly less than δ, which is the maximum welfare level in the circle. In fact, this result holds for any recognition probability p, as long as $p_2 > 0$ and $p_4 > 0$.

One can observe the negative welfare effect of adding a new link by computing the expected periods for a unanimous agreement. See Figure 3. In the circle, it takes exactly 2 periods for a grand-coalition in the equilibrium. Note that all the players fully use their communication links whenever they are recognized as a proposer. In the chordal network, however, the expected periods for a unanimous agreement is 2.5.[8] If the even players are recognized as a proposer in the first period, then they chooses one of the odd players as a bargaining partner to induce a three-player circle. In the circle, grand-coalition immediately forms. However, if the odd players are initially recognized as a proposer, they induces a three-player chain. In the chain, then the leaf players decline to make an offer and hence an additional delay occurs with positive probability.

Remark. In the Braess's paradox with the original traffic network context, all the players are worse off; while in this bargaining game in a communication network, some players may be better off even though overall performance deteriorates.

Remark. In the random-proposer bargaining model, the equilibrium may not be unique even in the class of stationary subgame perfect equilibria.[9] However, the equilibrium constructed in Example 3, Example 4, and Example 5 is unique in the class of *symmetric* cutoff-strategy equilibria, in which identical players in terms of a position in a network and a recognition probability play the identical cutoff strategy.

5 Concluding Remarks

We introduce a new non-cooperative coalitional bargaining model for network-restricted environments and show that strategic delay is prevalent. If the underlying network is either complete or circular, there exists an efficient equilibrium no matter what the discount factor is. For any incomplete and non-circular network, however, if the discount factor is greater than a certain level, then players strategically cause a delay and inefficiency occurs.

It is worth noting that inefficiency occurs for high discount factors but it is still *asymptotically efficient*. As the discount factor increases over a certain threshold, delay occurs more and more frequently but it becomes less and less costly and hence the inefficiency eventually disappears as the discount factor converges to one. On the other hand, if the discount factor is low enough, then any equilibrium must be efficient no matter what the underlying network is, because the impatient players try to reach the agreement as soon as possible. In sum, the efficiency loss occurs if the discount factor is strictly greater than

[8] $$(p_2+p_4)\times 2 + (p_1+p_3)\Big[(p_1+p_3)\times 2 + (p_2+p_4)\Big((p_1+p_3)\times 3 + (p_2+p_4)\big((p_1+p_3)\times 4 + \cdots\big)\Big)\Big]$$
$$= \tfrac{1}{2}\times 2 + \tfrac{1}{4}\times 2 + \tfrac{1}{8}\times 3 + \tfrac{1}{16}\times 4 + \cdots$$
$$= 1 + \sum_{k=2}^{\infty} k\frac{1}{2^k} = 2.5.$$

[9] To overcome multiplicity of equilibria, the uniqueness of equilibrium payoffs has been studied in the random-proposer bargaining model. Eraslan (2002) shows the equilibrium payoff uniqueness for a weighted majority game and Eraslan and McLennan (2013) generalizes this result to a general simple game using fixed point index theorem. Unfortunately, those results cannot be applied to the model in which a player has a buyout option, because a player can expect some partial payoff by forming an intermediate subcoalition and hence the actual characteristic function that the players play is not of a simple game. The uniqueness of stationary equilibrium payoffs is conjectured in a broader class of characteristic function form games, but it still remains as an open question. See Eraslan and McLennan (2013) for a discussion.

the threshold but strictly less than 1. If the network is complete or circular, then there is no such a threshold and an efficient equilibrium exists for all discount factor.

In addition to investigating efficiency, analyzing the equilibrium payoff vector in the noncooperative model and comparing it with cooperative solution concepts are an important research agenda. The limiting equilibrium payoff vector in the model proposes a plausible *power index in networks* – for instance, it assigns $\frac{5}{12}$ to the dominating players and $\frac{1}{12}$ to the non-dominating players in a 4-player chordal network as in Example 5; while the Myerson-Shapley value assigns the same value to each player in any unanimity game no matter what the underlying network is (Myerson, 1977).[10] In this regard, it would be of interest as future research to develop an algorithm for finding an equilibrium payoff vector and to study its normative properties.

References

Abreu, D. and M. Manea (2012a): "Bargaining and efficiency in networks," *Journal of Economic Theory*, 147, 43–70.

——— (2012b): "Markov equilibria in a model of bargaining in networks," *Games and Economic Behavior*, 75, 1–16.

Aumann, R. J. and J. H. Dreze (1974): "Cooperative games with coalition structures," *International Journal of game theory*, 3, 217–237.

Braess, D. (1968): "Über ein Paradoxon aus der Verkehrsplanung," *Unternehmensforschung*, 12, 258–268.

Eraslan, H. (2002): "Uniqueness of stationary equilibrium payoffs in the Baron–Ferejohn model," *Journal of Economic Theory*, 103, 11–30.

Eraslan, H. and A. McLennan (2013): "Uniqueness of stationary equilibrium payoffs in coalitional bargaining," *Journal of Economic Theory*, 148, 2195–2222.

Gul, F. (1989): "Bargaining foundations of Shapley value," *Econometrica: Journal of the Econometric Society*, 81–95.

Hart, S. and A. Mas-Colell (1996): "Bargaining and value," *Econometrica: Journal of the Econometric Society*, 357–380.

Lee, J. (2015): "Bargaining and Buyout," *working paper*, available at http://www.research.ed.ac.uk/portal/files/21741650/BargainingBuyout.pdf.

Manea, M. (2011a): "Bargaining in dynamic markets with multiple populations," *working paper*.

——— (2011b): "Bargaining in stationary networks," *The American Economic Review*, 101, 2042–2080.

Myerson, R. B. (1977): "Graphs and cooperation in games," *Mathematics of operations research*, 2, 225–229.

Nguyen, T. (2015): "Coalitional bargaining in networks," *Operations Research*.

[10] This is because only the grand-coalition generate a positive surplus and all the players have the same marginal contribution in a unanimity game.

Appendices

A Proof of Lemma 3

If $|N(g)| = 2$, then the statement is obviously true. As an induction hypothesis, suppose the statement is true for any less-than-n-player game. Consider g with $|N(g)| = n$. For any $\pi \in \Pi(g)$, observe that summing (2) over N^π yields

$$\begin{aligned}\sum_{i\in N^\pi} u_i^\pi(\mathbf{x},\mathbf{q}) &= \sum_{i\in N^\pi} p_i^\pi \sum_{S\subseteq N^\pi} q_i^\pi(S)\left[e_i^\pi(S,\mathbf{x}) + \delta\left(\sum_{j\in S} x_j^\pi + \sum_{j\notin S} x_j^{\pi(i,S)}\right)\right] \\ &= \sum_{i\in N^\pi} p_i^\pi \sum_{S\subseteq N^\pi} q_i^\pi(S) X^{\pi(i,S)}, \end{aligned} \tag{6}$$

where $X^{\pi(i,N^\pi)} = 1$ and $X^{\pi(i,S)} = \delta \sum_{j\in N^{\pi(i,S)}} x_j^{\pi(i,S)}$ for all $S \subsetneq N^\pi$.

Sufficiency: Let $(\mathbf{x},\mathbf{q})$ is an efficient equilibrium. By the consistency condition, for all $S \subsetneq N^\pi$,

$$\sum_{j\in N^{\pi(i,S)}} x_j^{\pi(i,S)} = \sum_{j\in N^{\pi(i,S)}} u_j^{\pi(i,S)}(\mathbf{x},\mathbf{q}).$$

Since $(\mathbf{x},\mathbf{q})$ is efficient, the induction hypothesis and the definition of efficiency yield $X^{\pi(i,S)} = \delta\bar{u}(\Gamma^{\pi(i,S)})$ for all $S \subsetneq N^\pi$. Suppose for contradiction that there exists $\pi \in \Pi_{\mathbf{x},\mathbf{q}}(g)$, $i \in N^\pi$, and $S, S' \subseteq N_i^\pi$ such that $q_i^\pi(S) > 0$ and $\bar{u}(\Gamma^{\pi(i,S)}) < \bar{u}(\Gamma^{\pi(i,S')})$. Then i can strictly improve the sum of the players' payoff by putting more weight on S' in his coalition formation strategy and hence q_i^π cannot be a part of an efficient equilibrium.

Necessity: Given g, $\pi \in \Pi(g)$, and $(\mathbf{x},\mathbf{q})$, define a partial strategy profile $(\mathbf{x}_{|\pi},\mathbf{q}_{|\pi}) = \{(x^{\pi'}, q^{\pi'})\}_{\pi'\in\Pi(g^\pi)}$. By induction hypothesis, for all $\pi \in \Pi_{\mathbf{x},\mathbf{q}}(g) \setminus \{\pi^\circ\}$, $(\mathbf{x}_{|\pi},\mathbf{q}_{|\pi})$ is an efficient equilibrium for a game with g^π. Consider the initial state. By (6), in order to maximize $\sum_{i\in N} u_i(\mathbf{x},\mathbf{q})$, each player i must maximize $\sum_{S\subseteq N} q_i(S) X^{(i,S)}$. Since, for all $i \in N$ and all $S \in \mathcal{E}_i$, $(\mathbf{x}_{|(i,S)},\mathbf{q}_{|(i,S)})$ is an efficient equilibrium for a game with $g^{(i,S)}$, the condition $q_i \in \Delta(\mathcal{E}_i)$ maximizes $\sum_{i\in N} u_i(\mathbf{x},\mathbf{q})$ and hence $(\mathbf{x},\mathbf{q})$ is efficient. □

B Proof of Proposition 1

Proof of Part i).

Case 1: $|N(g)| = 2$. Let $N(g) = \{i, j\}$ and $p = (p_i, p_j)$ with $p_i + p_j = 1$. We show that a cutoff strategy profile $(\{p_1, p_2\}, \{q_i(N) = 1, q_j(N) = 1\})$ is an equilibrium by verifying player i has no profitable deviation strategy given player j's cutoff strategy. Note that player i's expected payoff from following her cutoff strategy is $p_i(1 - \delta p_j) + p_j(\delta p_i) = p_i$. First, consider player i's proposal strategy. Either making an offer with $y_j < \delta p_j$ or declining to make an offer yields an expected payoff δp_i. Making an offer with $y_j > \delta p_j$ is not profitable since the offer $y_j = \delta p_j$ will be accepted. Thus, player i cannot be better off by deviating from the given proposal strategy. Next, consider player i's response strategy. By rejecting any offer, player i expects the payoff p_i in the next period. Thus, rejecting any offer with $y_i < \delta p_i$ is optimal. It is clear that accepting any offer with $y_i \geq \delta p_i$ is optimal. Therefore, player i has no profitable deviation strategy given player j's cutoff strategy.

Case 2: $|N(g)| > 2$. Suppose that, for any game (g', p', δ) with $|N(g')| < |N(g)|$, $(\mathbf{p}', \bar{\mathbf{q}}')$ is an equilibrium, where $\mathbf{p}' = \{\{p_i'^\pi\}_{i\in N^\pi}\}_{\pi\in\Pi(g')}$ and $\bar{\mathbf{q}}' = \{\{\bar{q}_i^\pi\}_{i\in N^\pi}\}_{\pi\in\Pi(g')}$. Note that, in such an equilibrium, for each $i \in N(g')$, player i's expected payoff is p_i'. We show that a cutoff strategy profile $\sigma = (\mathbf{p}, \bar{\mathbf{q}})$ is an equilibrium for (g, p, δ) by verifying player i has no profitable deviation strategy given other players cutoff strategies. Recall that if player i follows the cutoff strategy, then her expected payoff is $p_i(1-\delta) + \delta p_i = p_i$. Since all the other players except for i are supposed to play stationary strategies, it is enough to consider the proposal strategy and the response strategy of player i separately.

- *Proposal strategy:* Consider player i's proposal strategy q_i such that $q_i(S) > 0$ for some $S \subsetneq N$ instead of $\bar{q}_i$. By forming $S \subsetneq N$, player i expects $p_i^{(i,S)}$ in the subsequent game, because $(g^{(i,S)}, p^{(i,S)}, \delta)$ is a less-than-n-player game with a complete network. In order for S to form, it must be $y_j \geq \delta p_j$ for all $j \in S \setminus \{i\}$. Note also that $p_i^{(i,S)} \leq p_S$.[11] Thus, player i's proposal gain from S-formation is

$$\delta p_i^{(i,S)} - \sum_{j \in S \setminus \{i\}} y_j \leq \delta p_S - \sum_{j \in S \setminus \{i\}} \delta p_j = \delta p_i. \tag{7}$$

  On the other hand, player i's proposal gain from following $\bar{q}_i$ is

$$1 - \sum_{j \in N \setminus \{i\}} \delta p_j = (1-\delta) p_N + \delta p_i = (1-\delta) + \delta p_i. \tag{8}$$

  Since (7) is strictly less than (8), any proposal strategy which forms $S \subsetneq N$ is not optimal for i. Among proposal strategies which form N, it is clear that making an offer with $y = \delta p$ is optimal.

- *Response strategy:* Since each $j \in N \setminus \{i\}$ is supposed to play the given cutoff strategy, player i is guaranteed at least δp_i by rejecting any offer. Thus, it is optimal for i to accept any offer with $y_i \geq \delta p_i$ and to reject any offer with $y_i < \delta p_i$. □

Proof of Part ii).
The statement is true for $|N(g)| = 2$ from the proof of Proposition 1. As an induction hypothesis, suppose that the statement is true for any game with less-than-n-player games and now consider a game $\Gamma = (g, p, \delta)$ with $|N(g)| = n$. Due to Lemma 1, only cutoff strategy equilibria are considered. Suppose that there exists a cutoff strategy equilibrium $(\mathbf{x}, \mathbf{q})$.
Case 1: Suppose that $q = \bar{q}$. For each $i \in N$, since $\sum_{k \in N} p_k \sum_{S \subseteq N} q_k(S) \mathbb{1}(i \in S) = 1$, we have

$$u_i(\mathbf{x}, \bar{\mathbf{q}}) = p_i (1 - \delta x_N) + \delta x_i. \tag{9}$$

Due to consistency, we obtain $u_i(\mathbf{x}, \bar{\mathbf{q}}) = x_i$ and $x_N = 1$. Plugging them into (9), we have $x_i = p_i$. Thus, for any cutoff equilibrium involving maximum coalition formation strategies $\bar{q}$ yields a payoff vector p.
Case 2: Suppose that there exists i who plays a non-maximum coalition formation strategy so that $q_i(S) > 0$ with $S \subsetneq N$. This implies that

- $x_N = u_N(\mathbf{x}, \mathbf{q}) < 1$; and
- there exists $S \subsetneq N$ such that $i \in S$ and $e_i(S, \mathbf{x}) \geq e_i(N, \mathbf{x})$.

Thus for each $i \in S$, we have

$$\delta x_i^{(i,S)} - \delta x_S \geq 1 - \delta x_N > 1 - \delta.$$

[11] With transferable recognition probabilities, it holds with equality. With non-transferable recognition probabilities, the inequality is strict.

By the induction hypothesis, the inequality implies

$$\delta x_S + 1 < \delta p_S + \delta. \tag{10}$$

On the other hand, by letting $Q_j = \sum_{k\in N} p_k \sum_{S\subseteq N} q_k(S)\mathbb{1}(j \in S)$, for each $j \in S$, we have

$$\begin{aligned} x_j = u_j(\mathbf{x}, \mathbf{q}) &\geq p_j\,(1 - \delta x_N) + \delta(Q_j x_j + (1 - Q_j)p_j) \\ &> p_j\,(1 - \delta) + \delta(Q_j x_j + (1 - Q_j)p_j) \\ &= p_j + \delta Q_j(x_j - p_j). \end{aligned} \tag{11}$$

Rearranging the terms, (11) yields $x_j > p_j$ for all $j \in S$. However, this contradicts to (10) for all δ. □

C Proof of Proposition 2

Define $\eta(g) = \lfloor |N(g)|/2 \rfloor - 1$. If g is circular and $\eta(g) = 0$, then g must be a three-player circle, which is complete. Proposition 1 proves this case. As an induction hypothesis, suppose that, for all circular network g' such that $\eta(g') < m$, a cutoff strategy profile $(\mathbf{x}', \bar{\mathbf{q}}')$ is an equilibrium for (g', p', δ), where $\mathbf{x}' = \{\{\delta^{\eta(g'^{\pi})} p_i'^{\pi}\}_{i\in N(g'^{\pi})}\}_{\pi\in\Pi(g')}$. Now we show that a cutoff strategy profile $(\mathbf{x}, \bar{\mathbf{q}})$ is an equilibrium for (g, p, δ) with a circular network g and $\eta(g) = m$, where $\mathbf{x} = \{\{\delta^{\eta(g^{\pi})} p_i^{\pi}\}_{i\in N(g^{\pi})}\}_{\pi\in\Pi(g)}$. Take any $i \in N$ and let $N_i = \{i, j, k\}$. We verify the equilibrium conditions for player i.

i) *Optimality:* After i's maximum coalition formation, the active players face a game with a circular network g' and $\eta(g') = m - 1$. Due to the induction hypothesis, since $x_i^{(i,\{i,j,k\})} = \delta^{m-1}(p_i + p_j + p_k)$, we have

$$\begin{aligned} e_i(\{i, j, k\}, \mathbf{x}) &= \delta^m(p_i + p_j + p_k) - \delta(x_i + x_j + x_k) \\ &= \delta^m(p_i + p_j + p_k) - \delta(\delta^m p_i + \delta^m p_j + \delta^m p_k) \\ &= \delta^m(1 - \delta)(p_i + p_j + p_k). \end{aligned} \tag{12}$$

Suppose i decline to make an offer, that is i forms $\{i\}$. Since $e_i(\{i\}, \mathbf{x}) = 0$ is strictly less than (12), i's $\{i\}$-formation is not optimal. Suppose i forms $\{i, j\}$. Note that

$$x_i^{(i,\{i,j\})} = \begin{cases} \delta^{m-1}(p_i + p_j) & \text{if } |N(g)| \text{ is even,} \\ \delta^m(p_i + p_j) & \text{if } |N(g)| \text{ is odd.} \end{cases}$$

Thus, we have

$$e_i(\{i, j\}, \mathbf{x}) \leq \delta^m(p_i + p_j) - \delta(x_i + x_j) = \delta^m(1 - \delta)(p_i + p_j),$$

which is strictly less than (12), and hence i's S-formation with $|S| = 2$ is not optimal.

ii) *Consistency:* Since all the players play maximum coalition formation strategies, player i's continuation payoff is:

$$\begin{aligned} u_i(\mathbf{x}, \mathbf{q}) &= p_i e_i(\{i, j, k\}, \mathbf{x}) + \delta\left((p_i + p_j + p_k)x_i + \sum_{\ell\in N\setminus\{i,j,k\}} p_\ell x_i^{(\ell,N_\ell)}\right) \\ &= p_i\delta^m(1 - \delta)(p_i + p_j + p_k) + \delta(p_i + p_j + p_k)x_i + \delta(1 - (p_i + p_j + p_k))\delta^{m-1}p_i. \end{aligned}$$

Since $u_i(\mathbf{x},\mathbf{q}) = x_i$, rearranging the terms, we have

$$(1-\delta(p_i+p_j+p_k))x_i = p_i\delta^m(1-\delta)(p_i+p_j+p_k) + (1-(p_i+p_j+p_k))\delta^m p_i,$$

which yields $x_i = \delta^m p_i$. □

D Proof of Proposition 3

In a pre-complete network, there may or may not be exist a dominating player. We divide the proof into two disjoint cases, $D(g) \neq \emptyset$ and $D(g) = \emptyset$.

Case 1: $D(g) \neq \emptyset$

Since g is a pre-complete, note that there exists j_1 and j_2 such that $d(j_1, j_2; g) = 2$. Let $J_1(g) = N_{j_1}(g)\setminus D(g)$, $J_2(g) = N_{j_2}(g)\setminus D(g)$, and $J(g) = J_1(g)\cup J_2(g)$. Lemma 4 provides a lower bound of the unique dominating player's expected payoff.

Lemma 4. *Let g be a pre-complete network with $D(g) = \{i\}$. If $(\mathbf{x},\mathbf{q})$ is an equilibrium of $\Gamma = (g,p,\delta)$, then*

$$x_i \geq p_i + p_i(1-p_i)\delta. \tag{13}$$

Proof. Step 1: Consider a three-person chain, that is, $J_1 = \{j_1\}$ and $J_2 = \{j_2\}$. Since $x_i^{(j_1,J_1)} = x_i^{(j_2,J_2)} = x_i$ and $u_N(\mathbf{x},\mathbf{q}) \leq \bar{u}(\Gamma) = p_i + \delta(1-p_i)$, player i's expected payoff is

$$\begin{aligned} x_i &\geq p_i e_i(N,\mathbf{x}) + \sum_{k\in N} p_k \sum_{S\ni i} q_k(S)\delta x_i + \delta \sum_{k\in N} p_k \sum_{S\not\ni i} q_k(S) x_i \\ &\geq p_i(1-\delta(p_i+\delta(1-p_i))) + \delta x_i. \end{aligned}$$

Rearranging the terms, we have the desired result.

Step 2: As an induction hypothesis, assume that for any pre-complete network g' with $D(g') = \{i\}$, $\leq |J_1(g')| \leq a$, and $1 \leq |J_2(g')| \leq b$, $x_i' \geq p_i' + p_i'(1-p_i')\delta$. Now we consider a pre-complete network g with $D(g) = \{i\}$, $|J_1(g)| = a$, and $|J_1(g)| = b+1$. Player i's expected payoff is

$$x_i \geq p_i e_i(N,\mathbf{x}) + \sum_{k\in N} p_k \sum_{S\ni i} q_k(S)\delta x_i + \delta \sum_{k\in N} p_k \sum_{S\not\ni i} q_k(S) x_i^{(k,S)}. \tag{14}$$

For any $k \in N$ and $S \subseteq N$ such that $i \notin S$, the induction hypothesis implies $x_i^{(k,S)} \geq p_i + p_i(1-p_i)\delta$. Suppose by way of contradiction that $p_i + p_i(1-p_i)\delta > x_i$. Then, (14) can be written as $x_i \geq p_i\left(1-\delta(p_i+\delta(1-p_i))\right) + \delta x_i$, or equivalently, $x_i > p_i + p_i(1-p_i)\delta$, which yields a contradiction. Similarly, induction argument completes the proof. □

PROOF OF PROPOSITION 3 (CASE 1: $D(g) \neq \emptyset$)
Take any $j \in J_1$. Since $(\mathbf{x},\mathbf{q})$ is efficient, we have $(\forall j' \in J_2)\ \sum_{S\in\mathcal{C}_{j'}} q_{j'}(S) = 1$ and $(\forall i \in D\cup J_1)\ \sum_{j\in S\subseteq N} q_i(S) = 1$. Thus, player j's payoff is

$$\begin{aligned} u_j(\mathbf{x},\mathbf{q}) &= p_j m_j(\mathbf{x}) + \delta(p_D + p_{J_1})x_j + \delta \sum_{j'\in J_2} p_{j'} \sum_{S\subseteq N} q_{j'}(S) x_j^{(j',S)} \\ &\geq \delta(p_D + p_{J_1})x_j + \delta p_{J_2} p_j, \end{aligned}$$

which implies that $x_j \geq \frac{p_j p_{J_2} \delta}{1-(1-p_{J_2})\delta}$. Summing j over J_1, we have $x_{J_1} \geq \frac{p_{J_1} p_{J_2} \delta}{1-(1-p_{J_2})\delta}$. Similarly for J_2, we have $x_{J_2} \geq \frac{p_{J_1} p_{J_2} \delta}{1-(1-p_{J_1})\delta}$, and hence

$$x_J = x_{J_1} + x_{J_2} \geq p_{J_1} p_{J_2} \delta \left(\frac{1}{1-(1-p_{J_1})\delta} + \frac{1}{1-(1-p_{J_2})\delta} \right) \tag{15}$$

Now take any $i \in D$. Player i's optimality implies $e_i(N, \mathbf{x}) \geq e_i(D, \mathbf{x})$, or equivalently, $1 - \delta x_N \geq \delta x_i^{(i,D)} - \delta x_D$. Since $g^{(i,D)}$ has a single dominating player, Lemma 4 implies $x_i^{(i,D)} \geq p_D + p_D(1-p_D)\delta$ and it follows that

$$1 - p_D \delta (1 + \delta - p_D \delta) \;\geq\; \delta x_J \tag{16}$$

By (15) and (16), we have

$$1 - p_D \delta (1 + \delta - p_D \delta) \;\geq\; p_{J_1} p_{J_2} \delta^2 \left(\frac{1}{1-(1-p_{J_1})\delta} + \frac{1}{1-(1-p_{J_2})\delta} \right). \tag{17}$$

As $\delta \to 1$, the right hand side of (17) converges to p_J; while the left hand side converges to p_J^2. Since $p_J < 1$, there exists $\bar{\delta} < 1$ such that the inequality (17) yields a contradiction for $\delta > \bar{\delta}$. □

Case 2: $D(g) = \emptyset$

Before proving this, some lemmas are presented. First, whenever there is a dominating player in an incomplete network, Lemma 5, Lemma 6, and Lemma 7 show dominating players have some additional bargaining power compared to other non-dominating players. In a network without a dominating player, Lemma 8 shows that each player's payoff should be strictly less than her recognition probability under any efficient equilibrium. For any non-circular pre-complete network without a dominating player, Lemma 9 finds a player who can be a dominating player avoiding a complete network. Combining those lemmas, therefore, when an efficient equilibrium is assumed, at least one player can be strictly better off by strategically delaying a unanimous agreement, which is a contradiction in turn.

More graph-theoretic definitions are required. A *complete cover* of g is a collection $\mathcal{M}$ of subsets of $N(g)$, such that, $\cup\mathcal{M} = N(g)$ and $g_{|M}$ is a complete network for all $M \in \mathcal{M}$. A *complete covering number* of g is the minimum cardinality of a complete cover of g. A *minimal complete cover* is a complete cover of which cardinality is minimum.

Lemma 5. *Let g be a pre-complete network with $\emptyset \subsetneq D(g) \subsetneq N(g)$ and $(\mathbf{x}, \mathbf{q})$ be an equilibrium of (g, p, δ) with $\delta < 1$. If $i \in D(g)$ then $x_i > p_i$.*

Proof. If $|N(g)| = 3$, due to Lemma 4, then $x_i \geq p_i + p_i(1-p_i)\delta > p_i$ for any $i \in D$. As an induction hypothesis, suppose the statement is true for any g' with $|N(g')| < n$. Now consider g with $|N(g)| = n$. Take any $i \in D(g)$. For any $k \in N$ and any S such that $i \notin S$, if $g^{(k,S)}$ is complete then $x_i^{(k,S)} = p_i$; and if $g^{(k,S)}$ is incomplete then $x_i^{(k,S)} > p_i$ by the induction hypothesis. Thus, letting $Q_i = \sum_{k \in N} p_k \left(\sum_{S \ni i} q_i(S) + q_k(\{k\}) \right)$, we have $x_i \geq p_i(1 - \delta x_N) + Q_i \delta x_i + \delta(1 - Q_i) p_i$, and hence $x_i \geq p_i + \frac{\delta(1-\delta)}{1-\delta Q_i} p_i > p_i$. □

Lemma 5 says that for any dominating player, her expected payoff is strictly greater than her recognition probability. However, we need a stronger result: the difference between the expected payoff and the recognition probability is strictly positive even in the limit of that the discount factor converges to one. Lemma 6 shows that there exists such a dominating player and Lemma 7 proves it for all dominating players. For notational convenience, denote $\Delta_i = x_i - p_i$ and $\Delta_i^{(j,S)} = x_i^{(j,S)} - p_i^{(j,S)}$. If $g^{(i,S)}$ is complete and non-trivial, by Proposition 1, note that $e_i(S, \mathbf{x}) = \delta(x_i^{(i,S)} - x_S) = \delta(p_S - x_S) = -\delta\Delta_S$.

Lemma 6. *Let g be a pre-complete network with $\emptyset \subsetneq D(g) \subsetneq N(g)$ and $(\mathbf{x}, \mathbf{q})$ be an equilibrium of (g, p, δ). There exists $h \in D(g)$ such that*

$$x_h - p_h \geq \frac{p_h \left(p_D(1-p_D)\delta^2 - (1-\delta)\right)}{1 + (|D|-1)p_h\delta}.$$

Furthermore, $\lim_{\delta \to 1}(x_h - p_h) \geq \frac{p_h p_D(1-p_D)}{1+(|D|-1)p_h} > 0$

Proof. Take any $h \in \operatorname{argmax}_{i \in N} \Delta_i$ and let $Q_h = \sum_{i \in N} \sum_{S \ni h} p_i q_i(S)$. For any $i \in N$ and $S \subseteq N$ such that $h \notin S$, since $h \in D(g^{(i,S)})$, Lemma 5 implies $x_h^{(i,S)} \geq p_h$, and hence we have

$$\begin{aligned} x_h &\geq p_h e_h(D, \mathbf{x}) + Q_h \delta x_h + \delta(1-Q_h)p_h \\ &\geq p_h p_D(1-p_D)\delta^2 - p_h \Delta_D \delta + Q_h \delta(p_h + \Delta_h) + (1-Q_h)\delta p_h \\ &\geq p_h p_D(1-p_D)\delta^2 - p_h |D| \Delta_h \delta + \delta p_h + p_h \Delta_h \delta, \end{aligned} \tag{18}$$

where the second inequality is due to Lemma 4, which implies

$$e_h(D, \mathbf{x}) = \delta\left(x_h^{(h,D)} - x_D\right) \geq \delta(p_D + p_D(1-p_D)\delta - x_D) = p_D(1-p_D)\delta^2 - \Delta_D\delta,$$

and the last inequality comes from $p_h \leq Q_h$, $p_h \geq x_h$, and $\Delta_D \leq |D|\Delta_h$. Subtracting p_h from both sides of (18), we have $\Delta_h \geq \frac{p_h\left(p_D(1-p_D)\delta^2-(1-\delta)\right)}{1+(|D|-1)p_h\delta}$, as desired. Since $D \subsetneq N$, it must be $p_D < 1$ and hence $\lim_{\delta \to 1} \Delta_h \geq \frac{p_h p_D(1-p_D)}{1+(|D|-1)p_h} > 0$. ∎

Lemma 7. *Let g be a pre-complete network with $\emptyset \subsetneq D(g) \subsetneq N(g)$ and $(\mathbf{x}, \mathbf{q})$ be an equilibrium of (g, p, δ). For any $i \in D(g)$, there exists $\underline{\Delta}_i > 0$ such that $x_i - p_i \geq \underline{\Delta}_i$ as δ converges to 1.*

Proof. We will show that $\lim_{\delta \to 1} \min_{i \in D} \Delta_i > 0$. Let $L = \operatorname{argmin}_{i \in D} \Delta_i$. Since g is a pre-complete, as before there exists j_1 and j_2 such that $d(j_1, j_2; g) = 2$, and let $J_1(g) = N_{j_1}(g) \setminus D(g)$, $J_2(g) = N_{j_2}(g) \setminus D(g)$, and $J(g) = J_1(g) \cup J_2(g)$. Recall Lemma 5, which implies $(\forall i \in D)\ \Delta_i > 0$. Thus, for any $j \in J_1$ and $S \subsetneq N$, if $q_j(S) > 0$ then either $S \subseteq J_1$ or $S \cap D = \{\ell\}$ for some $\ell \in L$.

Case 1: Suppose $|J_1| = |J_2| = 1$. Then, for each $j \in J$, $q_j(\{j\}) + \sum_{\ell \in L} q_j(\{j, \ell\}) = 1$, and hence there exists $\ell \in L$ such that $\sum_{j \in J} p_j \left(q_j(\{j\}) + q_j(\{j, \ell\})\right) \geq \frac{p_J}{|L|}$. Let $Q_\ell = \sum_{j \in J} p_j \left(q_j(\{j\}) + q_j(\{j, \ell\})\right) + \sum_{i \in D} \sum_{S \ni \ell} p_i q_i(S)$, then $Q_\ell \geq \frac{p_J}{|L|} + p_\ell$. Since $x_\ell \geq p_\ell e_\ell(J \cup \{\ell\}, \mathbf{x}) + Q_\ell \delta x_\ell + (1-Q_\ell)\delta p_\ell$, it follows

$$\Delta_\ell \geq \delta p_\ell(\Delta_\ell + \Delta_J) + \delta\left(\frac{p_J}{|L|} + p_\ell\right)\Delta_\ell - (1-\delta)p_\ell,$$

which implies $\Delta_\ell \geq \frac{-\delta p_\ell \Delta_J - (1-\delta)p_\ell}{1-\delta\frac{p_J}{|L|}}$. Since $x_N - p_N = \Delta_N < 0$, we have $-\Delta_J \geq \Delta_D \geq \Delta_h$. Thus, by Lemma 6, we have the desired result,

$$\lim_{\delta \to 1} \Delta_\ell \geq -\frac{|L| p_\ell}{|L| - p_J}\Delta_J \geq \frac{|L| p_\ell}{|L| - p_J}\Delta_h \geq \frac{p_\ell p_h p_D(1-p_D)|L|}{(|L| - p_J)(1 + (|D|-1)p_h)} > 0.$$

Case 2: As an induction hypothesis, for any pre-complete network g' with $\emptyset \subsetneq D(g') \subsetneq N(g')$ and $1 \leq |J_1(g')| \leq a$ and $1 \leq |J_2(g')| \leq b$ and any equilibrium $(\mathbf{x}', \mathbf{q}')$ of (g', p', δ), assume that $\lim_{\delta \to 1} \min_{i \in D(g')}(x'_i - p'_i) > 0$. Now we consider a pre-complete network g with $\emptyset \subsetneq D(g) \subsetneq N(g)$ and $|J_1(g)| = a$ and $|J_2(g)| = b+1$. Due to the induction hypothesis, there exists $\Delta'_\ell > 0$ such that $\Delta'_\ell \geq \lim_{\delta \to 1}(x_\ell^{(j,J')} - p_\ell)$ for all $\alpha \in \{1,2\}$, $j \in J_\alpha$, and $J' \subseteq J_\alpha$. Then, we have

$$x_\ell \geq p_\ell e_\ell(J \cup \{\ell\}, \mathbf{x}) + \left(p_\ell + \sum_{\alpha \in \{1,2\}} \sum_{j \in J_\alpha} p_j \left(q_j(\{j\}) + q_j(J_\alpha \cup \{\ell\})\right) \right) \delta(p_\ell + \Delta_\ell)$$
$$+ \left(\sum_{\alpha \in \{1,2\}} \sum_{j \in J_\alpha} \sum_{J' \subseteq J_\alpha} p_j q_j(J') \right) \delta(p_\ell + \Delta'_\ell) + p_{D \setminus \{\ell\}} \delta p_\ell. \quad (19)$$

If $\lim_{\delta \to 1} \Delta_\ell \geq \Delta'_\ell$, then there is nothing to prove. Suppose that $\lim_{\delta \to 1} \Delta_\ell \leq \Delta'_\ell$. As $\delta \to 1$, then (19) yields $x_\ell \geq -p_\ell \delta \Delta_J + \delta p_\ell + (1 - p_D)\delta \Delta_\ell$, or equivalently, $(1 - (1 - p_D)\delta)\Delta_\ell \geq -\delta p_\ell \Delta_J - (1-\delta)p_\ell$. Take any $h \in \operatorname{argmax}_{i \in D} \Delta_i$. Since $-\Delta_J > \Delta_D > \Delta_h$, it follows that

$$(1 - (1 - p_D)\delta)\Delta_\ell > \delta p_\ell \Delta_h - (1-\delta)p_\ell.$$

By Lemma 5, we have the desired result, $\lim_{\delta \to 1} \Delta_\ell \geq \frac{p_\ell p_h (1 - p_D)}{1 + (|D|-1)p_h} > 0$. □

Lemma 8. *Let g be a pre-complete network with $D(g) = \emptyset$. If $(\mathbf{x}, \mathbf{q})$ is an efficient equilibrium of $\Gamma = (g, p, \delta)$, then for all $i \in N$, $x_i = \delta p_i$.*

Proof. Since g is pre-complete and $(\mathbf{x}, \mathbf{q})$ is efficient, for all $j \in N$, $q_j(S) > 0$ implies $g^{(j,S)}$ is complete. Thus, each player i can expect p_i in the next period by rejecting any offer. Suppose player i gets an offer with $y_i < \delta^2 p_i$. By rejecting y_i, i can be strictly better since the stationary strategy profile guarantees δp_i in the next period. Hence, $x_i \geq \delta p_i$ for all $i \in N$. If there exists $i \in N$ such that $x_i > \delta p_i$, then it must be $x_N > \delta p_N = \delta$, which is infeasible. □

Lemma 9. *Let g be a pre-complete non-circular network with $D(g) = \emptyset$. There exist $i, j \in N(g)$ such that $i \in D(g^{(i,\{i,j\})}) \subsetneq N(g^{(i,\{i,j\})})$.*

Proof. Since g is pre-complete non-circular, its complete covering number is 2. Let $\mathcal{M}$ be a minimal complete cover of g. Since $D(g) = \emptyset$, $\mathcal{M}$ must be disjoint. Given $i \in N$, then let $M_i \in \mathcal{M}$ such that $i \in M_i$. Since $D(g) = \emptyset$, for all $k \in N$, there exists at least one $k' \in M_k^c$ such that $kk' \notin E(g)$, that is, it must be $|M_k^c \setminus N_k(g)| \geq 1$. We will show that there exists $i \in N$ and $j \in M_i^c$ such that $i \in D(g^{(i,\{i,j\})}) \subsetneq N(g^{(i,\{i,j\})})$, by constructing such a pair of i and j in the following two cases. First, suppose there exists $k \in N$ such that $|M_k^c \setminus N_k(g)| \geq 2$. Take $i \in M_k^c \setminus N_k(g)$ and $j \in M_i^c$ with $ij \in E(g)$. Take $i' \in M_k^c \setminus N_k(g)$ with $i' \neq i$. Since $g_{|M_i}$ and $g_{|M_i^c}$ are complete, $i \in D(g^{(i,\{i,j\})})$. Since $d(k, i'; g) = d(k, i'; g^{(i,\{i,j\})}) = 2$, $k \notin N(g^{(i,\{i,j\})})$, as desired. Second, suppose, for all $k \in N$, $|M_k^c \setminus N_k(g)| = 1$. Take any $i \in N$ and $j \in M_i^c$ such that $ij \in E(g)$. Take $k \in M_i \setminus \{i\}$ and $k' \in M_i^c$ such that $d(k, k'; g) = 2$. Again we have $i \in D(g^{(i,\{i,j\})})$ and $d(k, k'; g) = d(k, k'; g^{(i,\{i,j\})}) = 2$, as desired. □

Proof of Proposition 3 (Case 2: $D(g) = \emptyset$)

Suppose $(\mathbf{x}, \mathbf{q})$ is an efficient equilibrium. Due to Lemma 8, for all $i \in N$ and all $S \in \mathcal{C}_i$,

$$e_i(S, \mathbf{x}) = \delta\left(x_i^{(i,S)} - x_S\right) = \delta\left(p_S - \delta p_S\right) = \delta(1-\delta)p_S,$$

which converges to 0 as $\delta \to 1$. By Lemma 9, there exists $i, j \in N(g)$ such that $i \in D(g^{(i,\{i,j\})})$ and $\{i,j\} \notin \mathcal{C}_i$. Due to Lemma 7, there exists $\underline{\Delta}_i$ such that $x_i^{(i,\{i,j\})} - p_i^{(i,\{i,j\})} \geq \underline{\Delta}_i$. By Lemma 8, then we have

$$\begin{aligned} e_i(\{i,j\}, \mathbf{x}) &= \delta \left(x_i^{(i,\{i,j\})} - (x_i + x_j) \right) \\ &\geq \delta \left((p_i^{(i,\{i,j\})} + \underline{\Delta}_i - \delta(p_i + p_j) \right) \\ &= \delta \underline{\Delta}_i + \delta(1-\delta)(p_i + p_j). \end{aligned}$$

As $\delta \to 1$, note that $e_i(\{i,j\}, \mathbf{x}) \geq \underline{\Delta}_i > 0$. Thus for a sufficiently high δ, $e_i(\{i,j\}, \mathbf{x}) > e_i(S, \mathbf{x})$ for all $S \in \mathcal{C}_i$, which contradicts to optimality of player i. □

E Proof of Proposition 4

Suppose that g is neither pre-complete nor complete. Now we construct a sequence of coalition formations which is consistent with $(\mathbf{x}, \mathbf{q})$ and the sequence induces a pre-complete network. Take $i^* \in \text{argmax}_{i \in N(g)} \deg_i(g)$. Let $I(g) = \{i \in N(g) \mid \mathcal{C}_i(g) = \emptyset\}$. Let $g_1 = g$ and take $i_1 \in \text{argmax}_{i \in I(g_1)} d(i, i^*; g_1)$. Pick any S_1 such that $q_{i_1}(S_1) > 0$. Let $g_2 = g^{(i_1, S_1)}$. Similarly, pick $i_2 \in \text{argmax}_{i \in I(g_2)} d(i, i^*; g_2)$. Pick any S_2 such that $q_{i_2}(S_2) > 0$. Since $(\mathbf{x}, \mathbf{q})$ is efficient, $|S_1| \geq 2$, $|S_2| \geq 2$, and so on; and $I(g_1) \supsetneq I(g_2) \supsetneq \cdots$. Thus, one can repeat this process until $I(g_T) = \emptyset$, after which g_T is a pre-complete network. This proves the first part. In addition, assume that g is not circular. If g is a tree, then any induced network cannot be circular and hence g_T is not circular. If g has a cycle but not a circular network, then $\deg_{i^*}(g) = \deg_{i^*}(g_T) \geq 3$, and hence g_T cannot be circular. □